Answers

PERTH COLLEGE
Year 12
Semester One Examination 2011
Question/Answer booklet

MATHEMATICS SPECIALIST 3CD

Section One: Calculator – free

Student Name: ______________________________

Time allowed for this section
Reading time before commencing work: 5 minutes
Working time for paper: 50 minutes

Material required/recommended for this section
To be provided by the supervisor
Question/answer booklet for Section One
Formula sheet

To be provided by the candidate
Standard items: pens, pencils, pencil sharpener, highlighter, eraser, ruler

Special items: nil

Important note to candidates
No other items may be taken into the examination room. It is **your** responsibility to ensure that you do not have any unauthorised notes or other items of a non-personal nature in the examination room. If you have any unauthorised material with you, hand it to the supervisor **before** reading any further.

Structure of this paper

	Number of questions available	Number of questions to be attempted	Working Time (minutes)	Marks available
Section One Calculator-free	**5**	**5**	**50 minutes**	**40**
Section Two Calculator-assumed	11	11	100 minutes	80
			Total marks	120

Instructions to candidates

1. Write your answers in the spaces provided in this Question/Answer Booklet. Spare pages are included at the end of this booklet. They can be used for planning your responses and/or as additional space if required to continue an answer
    a. Planning: If you use the spare pages for planning, indicate this clearly at the top of the page.
    b. Continuing an answer: If you need to use the space to continue an answer, indicate in the original answer space where the answer is continued, i.e. give the page number. Fill in the number of the question(s) that you are continuing to answer at the top of the page.

2. **Show all your working clearly.** Your working should be in sufficient detail to allow your answers to be checked readily and for marks to be awarded for reasoning. Incorrect answers given without supporting reasoning cannot be allocated any marks. For any question or part question worth more than two marks, valid working or justification is required to receive full marks. If you repeat an answer to any question, ensure that you cancel the answers you do not wish to have marked.

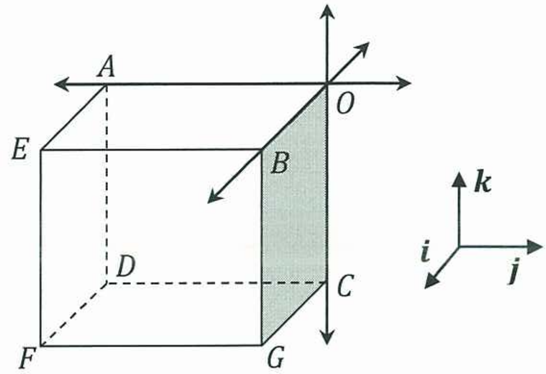
3. It is recommended that you **do not use pencil,** except in diagrams

Question 1 [1 + 2 + 2 = 5 marks]

In the diagram shown, $OAEBCDFG$ is a cube with the origin at O.

Let $\boldsymbol{i}$, $\boldsymbol{j}$ and $\boldsymbol{k}$ be the unit vectors along the x, y and z axes respectively, as shown.

Let $OA = \boldsymbol{a}$, $OB = \boldsymbol{b}$ and $OC = \boldsymbol{c}$.



(a) Find an expression for $\boldsymbol{a}, \boldsymbol{b}$ and $\boldsymbol{c}$ in terms of $\boldsymbol{i}, \boldsymbol{j}$ and/or $\boldsymbol{k}$, given that $|\boldsymbol{a}| = |\boldsymbol{b}| = |\boldsymbol{c}| = 2$ [1]

$a = -2j$

$b = 2i$ ✓

$c = -2k$

(b) Find OE and OF in terms of $\boldsymbol{i}, \boldsymbol{j}$ and $\boldsymbol{k}$. [2]

$\overrightarrow{OE} = -2j + 2i$ ✓

$\overrightarrow{OF} = -2k + 2i - 2j$ ✓

(c) Prove that FE and AB are perpendicular. [2]

$FE = 2k$ $\quad$ $AB = 2j + 2j$

$= \langle 0,0,2 \rangle$ $\quad$ $= \langle 2,2,0 \rangle$ ✓

$\langle 0,0,2 \rangle \cdot \langle 2,2,0 \rangle = 0$ ✓

$\therefore FE \perp AB$ QED

Question 2 [2 + 1 + 1 + 2 = 6 marks]

A plane Π is such that it is parallel to $\boldsymbol{i} + \boldsymbol{j}$, perpendicular to $\boldsymbol{k}$, and passes through $-(\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k})$.

(a) Find the vector equation of the plane Π. [2]

$\boxed{\text{parallel to } i + j}$

parallel to $i \to \langle 1,0,0 \rangle$

parallel to $j \to \langle 0,1,0 \rangle$

position vector $\langle -1,-1,-1 \rangle$

$r = \langle -1,-1,-1 \rangle + \lambda \langle 1,0,0 \rangle + \mu \langle 0,1,0 \rangle$ ✓✓

OR

$r = \langle -1,-1,-1 \rangle + \lambda \langle A,B,0 \rangle + \mu \langle C,D,0 \rangle$

where A,B,C,D are any numbers

(b) Find the normal equation of the plane Π.

* Find 3 points on the plane + work out 2 direction vectors for part (a) *

perpendicular to $k \to \langle 0,0,1 \rangle$

normal $= \langle 0,0,1 \rangle$

$r \cdot n = a \cdot n$

$r \cdot \langle 0,0,1 \rangle = \langle -1,-1,-1 \rangle \cdot \langle 0,0,1 \rangle$

$r \cdot \langle 0,0,1 \rangle = -1$ ✓

(c) Find the Cartesian equation of the plane Π. [1]

$\langle x,y,z \rangle \cdot \langle 0,0,1 \rangle = -1 \quad \therefore \quad z = -1$

(d) Find the point of intersection of the plane Π and the line $\boldsymbol{r} = (2\lambda + 1)(\boldsymbol{i} + \boldsymbol{j}) - \lambda \boldsymbol{k}$ [2]

$$r = \begin{pmatrix} 2\lambda + 1 \\ 2\lambda + 1 \\ -\lambda \end{pmatrix}$$

$$\therefore \begin{pmatrix} 2\lambda+1 \\ 2\lambda+1 \\ -\lambda \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = -1$$

$\therefore \lambda = 1$ ✓

$$\therefore \begin{pmatrix} 2(1)+1 = 3 \\ 2(1)+1 = 3 \\ -1 = -1 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \\ -1 \end{pmatrix}$$ ✓

Question 3 [3 + 2 + 3 + 2 + 3 = 13 marks]

Consider the following transformation matrices: $T_1 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ $T_2 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ $T_3 = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$

These transformation are performed in the order T_1, T_2 and then T_3.

(a) Describe the type of transformation performed by each matrix. [3]

T_1 = reflection over x axis ✓

T_2 = clockwise rotation 90° ✓

T_3 = horizontal shear factor of 2 ✓

(b) Find the matrix T_4 which will perform all three transformations in a single step. [2]

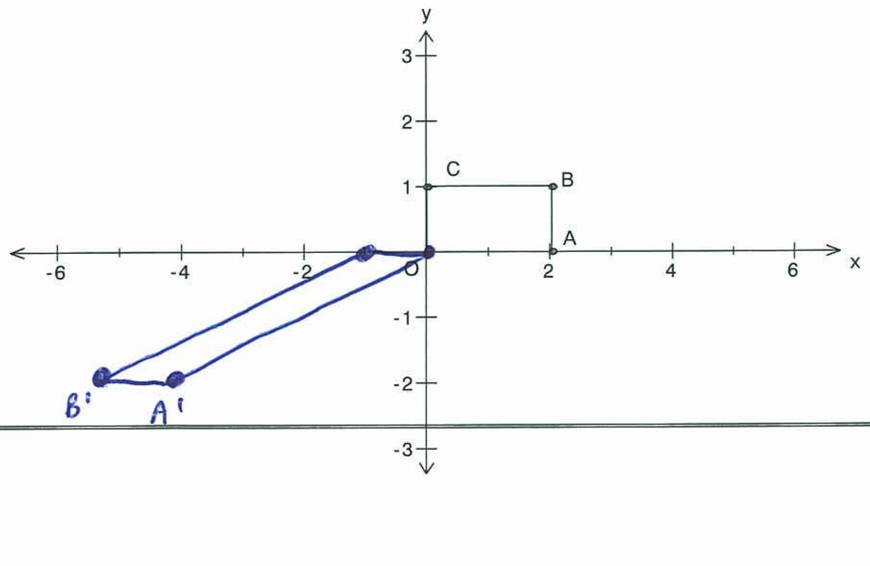
$T_4 = T_3 T_2 T_1$ ✓

$$T_4 = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} -2 & -1 \\ -1 & 0 \end{bmatrix} ✓$$

(c) Rectangle OABC is shown below. Find and sketch rectangle O'A'B'C' obtained using the transformation matrix found in (b). [3]

$$\begin{bmatrix} -2 & -1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 2 & 2 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} = \begin{array}{cccc} & A' & B' & C' \end{array} \begin{bmatrix} 0 & -4 & -5 & -1 \\ 0 & -2 & -2 & 0 \end{bmatrix} ✓✓$$



✓

(Question 3 continued)

(d) The matrix below contains the vertices O'M'N'P' of a shape transformed using T_4. Find the original vertices OMNP. [2]

$$\begin{bmatrix} 0 & -3 & 1 & 4 \\ 0 & 0 & 2 & 2 \end{bmatrix}$$

$$\begin{bmatrix} -2 & -1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} \ \end{bmatrix} = \begin{bmatrix} 0 & -3 & 1 & 4 \\ 0 & 0 & 2 & 2 \end{bmatrix}$$

$$\begin{bmatrix} \ \end{bmatrix} = \begin{bmatrix} -2 & -1 \\ -1 & 0 \end{bmatrix}^{-1} \begin{bmatrix} 0 & -3 & 1 & 4 \\ 0 & 0 & 2 & 2 \end{bmatrix} = -\frac{1}{1} \begin{bmatrix} 0 & 1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} 0 & -3 & 1 & 4 \\ 0 & 0 & 2 & 2 \end{bmatrix}$$

$$OMNP = \begin{bmatrix} 0 & 0 & -2 & -2 \\ 0 & 3 & 3 & 0 \end{bmatrix}$$

(e) The transformation matrix T_4 is applied to the line $y = 3 - x$. Prove that the new line obtained is perpendicular to the original line. [3]

$$\begin{bmatrix} -2 & -1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} p \\ 3-p \end{bmatrix} = \begin{bmatrix} -2p - 3 + p \\ -p \end{bmatrix}$$

$$= \begin{bmatrix} -p - 3 \\ -p \end{bmatrix}$$

$y^o = 3 - x$

$m^o = -1$

$$\therefore \left.\begin{aligned} x' &= -p - 3 \\ y' &= -p \end{aligned}\right\} \therefore y' = x' + 3$$

$M' = 1$ $\quad M^{\text{original}} = -1$ $\quad M' \times M^o = -1 \therefore \perp$

Question 4 [1 + 2 + 3 + 2 + 3 = 11 marks]

Find an expression for $\frac{dy}{dx}$ for each case below. Do not simplify your answers.

(a) $y = x^2 \sin x^2$ [1]

$y' = 2x \sin x^2 + (x^2)(2x) \cos x^2$ ✓

(b) $y = \dfrac{x^2 \ln x^2}{e^{x^2}}$ [2]

$$y' = \frac{\left[2x \ln x^2 + x^2 \cdot \frac{2x}{x^2}\right] e^{x^2} - \left[x^2 \ln x^2\right] 2x\, e^{x^2}}{\left[e^{x^2}\right]^2}$$ ✓✓

(c) $y = (2x)^{3x}$ (Hint: apply the natural logarithm on both sides) [3]

$\ln y = 3x \ln 2x$ ✓

$\frac{1}{y}\frac{dy}{dx} = 3 \ln 2x + 3x\left(\frac{2}{2x}\right)$ ✓

$\frac{dy}{dx} = y\left[3 \ln 2x + 3\right]$

$\frac{dy}{dx} = (2x)^{3x}\left[3 \ln 2x + 3\right]$ ✓

(Question 4 continued)

(d) $y = \ln t^2$, $x = e^t$ (leave your answer in terms of x) [2]

$$\frac{dy}{dt} = \frac{2t}{t^2} = \frac{2}{t} \qquad \frac{dx}{dt} = e^t$$

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} = \frac{2}{t} \times \frac{1}{e^t} = \frac{2}{te^t}$$

$$\therefore \frac{dy}{dx} = \frac{2}{te^t} \checkmark \qquad \frac{dy}{dx} = \frac{2}{x \ln x} \checkmark$$

$$x = e^t$$
$$\ln x = t \ln e$$
$$\ln x = t$$

(e) $x \sin y - y \sin x = \pi$ [3]

$$(1)\sin(y) + x\cos(y)\frac{dy}{dx} - \left[(1)\sin(x)\frac{dy}{dx} + y\cos(x)\right] = 0 \quad \checkmark \checkmark$$

$$\sin(y) + x\cos(y)\frac{dy}{dx} - \sin(x)\frac{dy}{dx} - y\cos(x) = 0$$

$$\frac{dy}{dx} = \frac{y\cos(x) - \sin(y)}{x\cos(y) - \sin(x)} \checkmark$$

Question 5 [5 marks]

Perform the following integration.

(Hint: Refer to formula sheet provided for a suitable trigonometric identity)

$$\int (\cos x)^3 \, dx$$

[5]

$= \int \cos(x) \cos^2(x) \, dx$ ✓

$= \int \cos(x) (1 - \sin^2(x)) \, dx$ ✓

$= \int \cos(x) - \cos(x) \sin^2(x) \, dx$

$= \int \cos(x) - \int \cos(x) \sin^2(x) \, dx$ ✓

$= \sin(x) - \dfrac{\sin^3(x)}{3} + c$

$y = \sin^3 x$

$\dfrac{dy}{dx} = 3\cos(x)\sin^2(x$

✓ ✓

END OF SECTION ONE

(Deliberately left blank)

(Deliberately left blank)

(Deliberately left blank)

PERTH COLLEGE
Year 12
Semester One Examination 2011
Question/Answer booklet

MATHEMATICS
SPECIALIST 3CD

Section Two
Calculator – assumed

Student Name: ______________________________

Time allowed for this section

Reading time before commencing work:	10 minutes
Working time for paper:	100 minutes

Material required/recommended for this section

To be provided by the supervisor
Question/answer booklet for Section Two
Formula sheet

To be provided by the candidate

Standard items: pens, pencils, pencil sharpener, highlighter, eraser, ruler

Special items: drawing instruments, templates, notes (both sides of two unfolded sheets of A4 paper) and up to three calculators (CAS, graphic or scientific) which satisfy the conditions set by the Curriculum Council for this course.

Important note to candidates

No other items may be taken into the examination room. It is **your** responsibility to ensure that you do not have any unauthorised notes or other items of a non-personal nature in the examination room. If you have any unauthorised material with you, hand it to the supervisor
before reading any further.

Structure of this paper

	Number of questions available	Number of questions to be attempted	Working time (minutes)	Marks available
Section One Calculator-free	5	5	50 minutes	40
Section Two **Calculator-assumed**	**11**	**11**	**100 minutes**	**80**
			Total marks	120

Instructions to candidates

1. Write your answers in the spaces provided in this Question/Answer Booklet. Spare pages are included at the end of this booklet. They can be used for planning your responses and/or as additional space if required to continue an answer
   a. Planning: If you use the spare pages for planning, indicate this clearly at the top of the page.
   b. Continuing an answer: If you need to use the space to continue an answer, indicate in the original answer space where the answer is continued, i.e. give the page number. Fill in the number of the question(s) that you are continuing to answer at the top of the page.

2. **Show all your working clearly.** Your working should be in sufficient detail to allow your answers to be checked readily and for marks to be awarded for reasoning. Incorrect answers given without supporting reasoning cannot be allocated any marks. For any question or part question worth more than two marks, valid working or justification is required to receive full marks. If you repeat an answer to any question, ensure that you cancel the answers you do not wish to have marked.

3. It is recommended that you **do not use pencil,** except in diagrams.

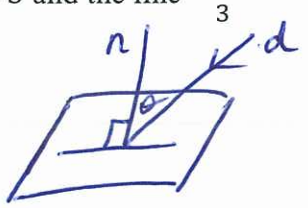
Question 6 [3 + 2 = 5 marks]

(a) Find the acute angle between the plane $r \cdot \begin{pmatrix} -2 \\ 1 \\ 3 \end{pmatrix} = 5$ and the line $\frac{x-1}{3} = \frac{y+1}{2} = \frac{2-z}{3}$. [3]

$\lambda = \frac{x-1}{3} \quad \therefore x = 3\lambda + 1$

$\lambda = \frac{y+1}{2} \quad \therefore y = 2\lambda - 1$

$\lambda = \frac{2-z}{3} \quad \therefore z = 3\lambda + 2$



$\cos\theta = \frac{n.d}{|n||d|}$

$r = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 2 \\ -3 \end{pmatrix}$ ← d

$\cos\theta = \frac{\langle -2, 1, 3 \rangle \cdot \langle 3, 2, -3 \rangle}{\sqrt{14} \cdot \sqrt{22}}$

$= \frac{-13}{\sqrt{308}}$ ✓

$\theta = 137.79°$ ✓

$\therefore$ acute value of $\theta = 42.21°$

$90° - 42.21° \Rightarrow$ $\therefore$ angle between line + plane is $47.79°$ ✓

(b) Find the value of k so that the vector $\begin{pmatrix} -k \\ k-2 \\ 8-k \end{pmatrix}$ is normal to the plane $3x = 12 + 2y + z$. [2]

$3x - 2y - z = 12$

$r . \langle 3, -2, -1 \rangle = 12 \qquad \therefore n = \langle 3, -2, -1 \rangle$

$\therefore \langle -k, k-2, 8-k \rangle = \lambda \langle 3, -2, -1 \rangle$

$3\lambda = -k$

$-2\lambda = k - 2$ ✓

$-\lambda = 8 - k$

$k = 6$ ✓

Question 7 [2 +3 + 2 + 5 = 12 marks]

(a) The line L_1 passes through the points $P(2, -1, 3)$ and $Q(-2, 3, 1)$.
Find the vector equation of line L_1. [2]

$\overrightarrow{PQ} = Q - P = \langle -2, 3, 1 \rangle - \langle 2, -1, 3 \rangle$
$= \langle -4, 4, -2 \rangle$ ✓

$L_1 \Rightarrow r = \langle 2, -1, 3 \rangle + \lambda \langle -4, 4, -2 \rangle$ ✓

(b) The line L_2 is perpendicular to line L_1 from (a), and intersects L_1 at $R(-2, 3, 1)$.
Find the vector equation of line L_2. [3]

$L_2 \Rightarrow r = a + \lambda d$
$r = \langle -2, 3, 1 \rangle + \lambda \langle a, b, c \rangle$

$\langle a, b, c \rangle \cdot \langle -4, 4, -2 \rangle = 0$ ✓ $L_1 \perp$ to L_2

any value of a, b, c that works
$a = 1 \quad b = 1 \quad c = 0$ ✓

$L_2 \Rightarrow r = \langle -2, 3, 1 \rangle + \lambda \langle 1, 1, 0 \rangle$ ✓

(c) Find equation of the plane Π that contains both L_1 and L_2 found above. [2]

$\Pi = \langle -2, 3, 1 \rangle + \lambda \langle -4, 4, -2 \rangle + \mu \langle 1, 1, 0 \rangle$

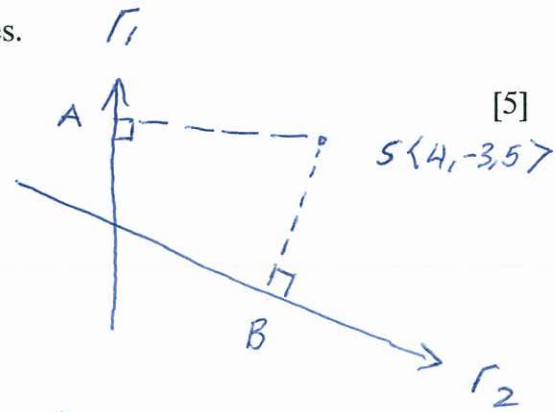
✓ ✓

OR OR

$r \cdot \langle 1, -1, -4 \rangle = -9$ SIMILAR

(Question 7 continued)

(d) Point $S(4,-3,5)$ is at a certain distance from both lines. Which line, L_1 or L_2, is the closest to point S? Show clear evidence to support your answer. [5]



$r_1 = \langle 2,-1,3\rangle + \lambda\langle -4,4,-2\rangle$

$r_2 = \langle -2,3,1\rangle + \lambda\langle 1,1,0\rangle$

$\vec{SA} = A - S = \langle 2-4\lambda, -1+4\lambda, 3-2\lambda\rangle - \langle 4,-3,5\rangle$

$= \langle -2-4\lambda, 2+4\lambda, -2-2\lambda\rangle$

$\vec{SA} \cdot (\text{dir } r_1) = 0$

$\langle -2-4\lambda, 2+4\lambda, -2-2\lambda\rangle \cdot \langle -4,4,-2\rangle = 0$

$\therefore 8+16\lambda+8+16\lambda+4+4\lambda = 0 \qquad \therefore \lambda = \frac{-5}{9}$

$\vec{SA} = \langle -2-4(\frac{-5}{9}), 2+4(\frac{-5}{9}), -2-2(\frac{-5}{9})\rangle = \langle \frac{2}{9}, -\frac{2}{9}, -\frac{8}{9}\rangle$ ✓

$|\vec{SA}| = \frac{2\sqrt{2}}{3} = 0.9428$ units ✓

$\vec{SB} = B - S = \langle -2+\lambda, 3+\lambda, 1\rangle - \langle 4,-3,5\rangle = \langle 6+\lambda, 6+\lambda, -4\rangle$

$\vec{SB} \cdot (\text{dir } r_2) = 0$

$\langle 6+\lambda, 6+\lambda, -4\rangle \cdot \langle 1,1,0\rangle = 0$

$-6+\lambda+6+\lambda = 0$

$\lambda = 0$

$\vec{SB} = \langle -6, 6, -4\rangle$ ✓

$|\vec{SB}| = \sqrt{88} = 9.38$ units ✓

L1 is closest to S ✓

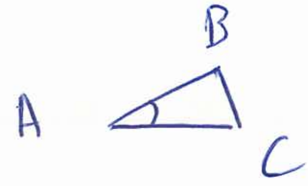
Question 8 [6 + 2 = 8 marks]

Triangle ABC has its vertices at $A(1,1,1)$, $B(2,1,2)$ and $C(0,2,1)$.

(a) Find the exact value of all internal angles of triangle ABC using the dot product. [6]

$\overrightarrow{AB} = B - A = \langle 2,1,2\rangle - \langle 1,1,1\rangle$

$= \langle 1,0,1\rangle$ ✓



$\overrightarrow{AC} = C - A = \langle 0,2,1\rangle - \langle 1,1,1\rangle = \langle -1,1,0\rangle$ ✓

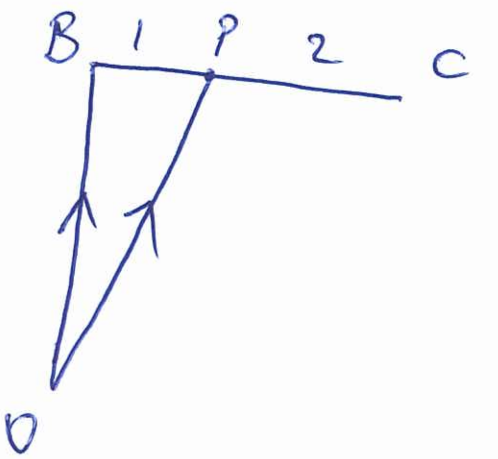
$|\overrightarrow{AB}| = \sqrt{2}$ $|\overrightarrow{AC}| = \sqrt{2}$ ✓

$\cos(A) = \dfrac{(AB)\cdot(AC)}{|AB|\,|AC|} = \dfrac{\langle 1,0,1\rangle \cdot \langle -1,1,0\rangle}{\sqrt{2}\cdot\sqrt{2}}$ ✓

$A = 120°$ ✓

$|\overrightarrow{AB}| = |\overrightarrow{AC}|$ ∴ isosceles ∴ $\angle C = \angle B = 30°$ ✓

(b) Point P divides the segment BC internally in the ratio $1:2$.
Find the coordinates of point P. [2]



$P = B + \frac{1}{3}\overrightarrow{BC}$

$P = B + \frac{1}{3}(C - B)$

$P = B + \frac{1}{3}C - \frac{1}{3}B$

$P = \frac{2}{3}B + \frac{1}{3}C$

$P = \frac{1}{3}(2B + C)$ ✓ ∴ $P = \frac{1}{3}\begin{pmatrix}4+0\\2+2\\4+1\end{pmatrix} = \frac{1}{3}\begin{pmatrix}4\\4\\5\end{pmatrix}$ ✓

Question 9 [2 + 3 + 3 = 8 marks]

(a) Solve: $|1 - x^2| \geq |x + 5|$ [2]

Classpad $x \leq -2$ ✓

$x \geq 3$ ✓

(b) Use an algebraic method to solve: $|2x + 1| \leq 2|1 - x|$ [3]

$2x + 1 \leq 2(1 - x)$

$2x + 1 \leq 2 - 2x$

$4x \leq 1$

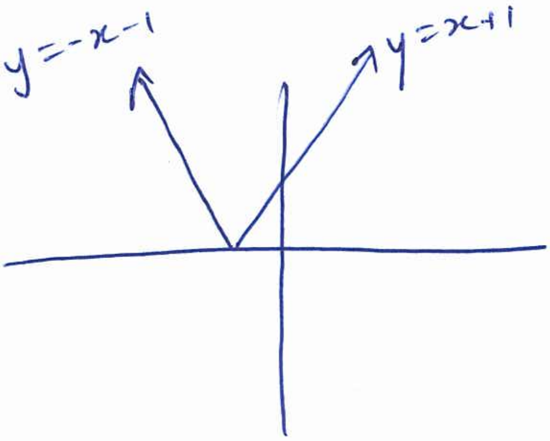
$x \leq \frac{1}{4}$ ✓

$2x + 1 \leq -2(1 - x)$ ✓

$2x + 1 \leq -2 + 2x$

No soln ✓

(c) Use a graphical or algebraic method to solve: $\sqrt{1 + |x|^2 + 2x} = |1 + x|$
Explain your answer. [3]



✓

Graphs are coincident ✓

$x \in \mathbb{R}$ ✓

OR

✓ $1 + |x|^2 + 2x = (1 + x)^2$

✓ $1 + x^2 + 2x = 1 + 2x + x^2$

$1 = 1$ ✓

true ALL $\mathbb{R}$ Value of x

Question 10 [4 marks]

Use first principles to show that the derivative of $y = (2x+1)^2$ is $\frac{dy}{dx} = 4(2x+1)$ [4]

$f(x) = (2x+1)^2$
$= 4x^2 + 4x + 1$

$f(x+h) = [2(x+h)+1]^2$
$= 4x^2 + 8xh + 4h^2 + 4x + 4h + 1$ ✓

$$\frac{dy}{dx} = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{(4x^2 + 8xh + 4h^2 + 4x + 4h + 1) - (4x^2 + 4x + 1)}{h}$$ ✓

$$= \lim_{h\to 0} \frac{8xh + 4h^2 + 4h}{h}$$

$$= \lim_{h\to 0} 8x + 4h + 4$$ ✓

$$= 8x + 4 = 4(2x+1)$$ ✓

Question 11 [6 marks]

A fish of species P each day consumes 8g of food A, 5g of food B and 3g of food C.
A fish of species Q each day consumes 5g of food A, 3g of food B and 2g of food C.
A fish of species R consumes 3g, 1g and 1g respectively of food A, B and C.
A given environment has 310g of food A, 170g of food B and 115g of food C.

(a) Use a matrix method to find the population size of the three species that will consume exactly all of the available food in:

(i) one day. [2]

$x \rightarrow$ Type P
$y \rightarrow$ Type Q
$z \rightarrow$ Type R

A $8x + 5y + 3z = 310$
B $5x + 3y + z = 170$
C $3x + 2y + z = 115$

$$\begin{bmatrix} 8 & 5 & 3 \\ 5 & 3 & 1 \\ 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 310 \\ 170 \\ 115 \end{bmatrix}$$

Type P = 20
Type Q = 15 ✓
Type R = 25

TOTAL 60 FISH ✓

(ii) five days. [2]

$$\frac{1}{5}\begin{bmatrix} 8 & 5 & 3 \\ 5 & 3 & 1 \\ 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 310 \\ 170 \\ 115 \end{bmatrix}$$

Type P = 4
Type Q = 3 ✓
Type R = 5

TOTAL 12 FISH ✓

(b) Given that there were 10 fish of each species in an environment, use a matrix method to determine how much food of each type would be required each day. [2]

$$\begin{bmatrix} 8 & 5 & 3 \\ 5 & 3 & 1 \\ 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} 10 \\ 10 \\ 10 \end{bmatrix} \checkmark = \begin{bmatrix} 160 \\ 90 \\ 60 \end{bmatrix}$$

160g of A 90g of B 60g of C ✓

Question 12 [5 marks]

Heat is applied to a spherical ball so that it expands while preserving its spherical shape.
When the surface area of the sphere is exactly π cm^2, it is increasing at the rate of 50 mm^2 per second.

Find the rate of increase of the volume of the ball at this instant. [5]

$SA = 4\pi r^2$ $V = \frac{4}{3}\pi r^3$

$\frac{dSA}{dt} = 50 mm^2/sec.$ at $SA = \pi cm^2$

Find $\frac{dV}{dt}$

$\frac{dSA}{dt} = 8\pi r \cdot \frac{dr}{dt}$

$50 = 8\pi r \cdot \frac{dr}{dt}$ $\therefore \frac{dr}{dt} = \frac{50}{8\pi r}$ ✓

$\frac{dV}{dt} = \frac{dV}{dr} \cdot \frac{dr}{dt}$

$\pi = 4\pi r^2$

$r^2 = \frac{1}{4}$ $\therefore r = \frac{1}{2}$ cm

$= 5$ mm ✓

$\therefore \frac{dr}{dt} = \frac{50}{8\pi(5)} = \frac{5}{4\pi}$

$\frac{dV}{dr} = 4\pi r^2$

$= 4 \times \pi \times 25 = 100\pi$ ✓

$\frac{dV}{dt} = \frac{dV}{dr} \cdot \frac{dr}{dt}$ ✓

$\frac{dV}{dt} = 100\pi \times \frac{5}{4\pi} = 125 mm^3/sec$ ✓

Question 13 [6 marks]

A curve is defined implicitly with the equation $y - x + 3e^{-y} = 1$.

Find the exact equation of the tangent to this curve at the point $(2, 0)$.

Show ALL working. [6]

$$(1)\frac{dy}{dx} - (1) + 3(-1)e^{-y}\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{1}{1-3e^{-y}}$$

at $y=0$ $\quad \frac{dy}{dx} = \frac{1}{1-3e^{-0}} = -\frac{1}{2}$

$$y = \frac{-1}{2}x + c$$

at $(2,0)$ $\quad 0 = -\frac{1}{2}(2) + c \quad \therefore c = 1$

$$\therefore y = \frac{-1}{2}x + 1$$

Question 14 [2 + 2 + 3 + 3 + 4 = 14 marks]

Evaluate the following integrals, showing sufficient working to justify your answers.

(a) $\displaystyle\int \frac{2x^2 + 3x}{\sqrt{x}}\, dx$ [2]

$= \displaystyle\int 2x^{3/2} + 3x^{1/2}\, dx$

$= \dfrac{2x^{5/2}}{5/2} + \dfrac{3x^{3/2}}{3/2} + c$

$= \dfrac{4}{5}x^{5/2} + 2x^{3/2} + c$ ✓✓

(b) $\displaystyle\int \frac{1}{\sqrt{1-2x}}\, dx$ [2]

$= \displaystyle\int (1-2x)^{-1/2}\, dx$

$= \dfrac{(1-2x)^{1/2}}{(-2)(1/2)} + c$ ✓ $= -(1-2x)^{1/2} + c$ ✓

(c) $\displaystyle\int \pi x\sqrt{1+x^2}dx$ [3]

Guess $y = (1+x^2)^{3/2}$

$\dfrac{dy}{dx} = \dfrac{3}{2}(1+x^2)^{1/2}(2x)$

$= 3x(1+x^2)^{1/2}$ ✓

adjust guess

✓ $= \dfrac{\pi}{3}(1+x^2)^{3/2}$ ✓ $+ c$

(Question 14 continued)

(d) $\int \frac{1}{2(1+\sqrt{x})\sqrt{x}}\,dx$ [3]

Let $u = 1+\sqrt{x}$

$\frac{du}{dx} = \frac{1}{2\sqrt{x}}$

$\frac{dx}{du} = 2\sqrt{x}$ ✓

$= \frac{1}{2}\int \frac{1}{(1+\sqrt{x})\sqrt{x}}\,dx$

$= \frac{1}{2}\int \frac{1}{u\sqrt{x}}\frac{dx}{du}\,du$

$= \frac{1}{2}\int \frac{1}{u\sqrt{x}}\cdot 2\sqrt{x}\,du$ ✓

$= \int \frac{1}{u}\,du = \ln|u| + c = \ln|1+\sqrt{x}| + c$ ✓

(e) $\int \frac{x+2}{(x+1)(x-2)}\,dx$ *(Hint: partial fractions)* [4]

$\frac{A}{(x+1)} + \frac{B}{(x-2)} = \frac{x+2}{(x+1)(x-2)}$

$A(x-2) + B(x+1) = x+2$

$Ax + Bx = x$

$-2A + B = 2$

$\therefore A+B = 1 \quad -2A+B = 2$

$A = -\frac{1}{3} \quad B = \frac{4}{3}$

$= \int \frac{-1}{3(x+1)}$ ✓ $+ \int \frac{4}{3(x-2)}\,dx$ ✓

$= -\frac{1}{3}\int \frac{1}{x+1} + \frac{4}{3}\int \frac{1}{x-2}\,dx$

$= -\frac{1}{3}\ln|x+1|$ ✓ $+ \frac{4}{3}\ln|x-2|$ ✓ $+ c$

Question 15 [1 + 3 + 2 + 3 = 9 marks]

Salmon can be bred in large ponds for commercial purposes.
A particular species of salmon lives to four years of age. The survival rate of this species in their first, second and third years is 0.5%, 7% and 15% respectively. It is known that females only reproduce during their third and fourth years of age. Each 3 year old female can produce 5 000 female offspring, while each 4 year old female can produce 2 000 female offspring.
A new pond is started with 1 000 of each 1, 2, 3 and 4 year old female salmon, including an appropriate amount of males.

(a) Find the Leslie matrix for this population. [1]

$$L = \begin{bmatrix} 0 & 0 & 5000 & 2000 \\ 0.005 & 0 & 0 & 0 \\ 0 & 0.07 & 0 & 0 \\ 0 & 0 & 0.15 & 0 \end{bmatrix}$$ ✓

(b) Showing working to support your answer, find (to the nearest whole number) the total female population after:

(i) 1 year [1]

$$\begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix} \times L \times \begin{bmatrix} 1000 \\ 1000 \\ 1000 \\ 1000 \end{bmatrix} = 7000225$$ people ✓

(ii) 2 years [1]

$$\begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix} \times L^2 \times \begin{bmatrix} 1000 \\ 1000 \\ 1000 \\ 1000 \end{bmatrix} = 685010.85$$
$= 685010$ people ✓

(iii) 5 years [1]

$$\begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix} \times L^5 \times \begin{bmatrix} 1000 \\ 1000 \\ 1000 \\ 1000 \end{bmatrix} = 1933792.61$$
$\therefore$ 1933792 female ✓

(Question 15 continued)

(c) When will the female population be more than four million again? [2]

$$[1\,1\,1\,1] \times L^4 \times \begin{bmatrix} 1000 \\ 1000 \\ 1000 \\ 1000 \end{bmatrix} = 12250813.75$$

✓

∴ 4 years later ✓

(d) After 6 years the salmon population is badly affected by disease, and hence the survival rates have decreased to 0.2%, 5% and 10% for the first, second and third years respectively. Determine whether the population of salmon is increasing or decreasing due to the disease. Show clear evidence to support your answer. [3]

$$L' = \begin{bmatrix} 0 & 0 & 5000 & 2000 \\ 0.002 & 0 & 0 & 0 \\ 0 & 0.05 & 0 & 0 \\ 0 & 0 & 0.1 & 0 \end{bmatrix}$$

$$P_{initial} = P_6 = \begin{bmatrix} 108062 \\ 9362 \\ 4287 \\ 1 \end{bmatrix}$$ ✓

$$[1\,1\,1\,1]\, L^{10} \cdot P_{in} = 2684792$$

$$[1\,1\,1\,1]\, L^{50} \cdot P_{in} = 264$$ ✓

$$[1\,1\,1\,1]\, L^{100}\, P_{in} = 0$$

Decreasing ✓

Question 16 [2 + 1 = 3 marks]

Consider matrix $A = \begin{bmatrix} x & 1+x \\ 1-x & -x \end{bmatrix}$

(a) Show that $A^2 = I$ [2]

$$\begin{bmatrix} x & 1+x \\ 1-x & -x \end{bmatrix} \begin{bmatrix} x & 1+x \\ 1-x & -x \end{bmatrix}$$

$$= \begin{bmatrix} x^2 + 1 - x^2 & x + x^2 - x - x^2 \\ x - x^2 - x + x^2 & 1 - x^2 + x^2 \end{bmatrix} \checkmark\checkmark$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

(b) State the restrictions on x, if any, for $A^2 = I$ [1]

$A^{-1} = A \Rightarrow A$ cannot be singular $\therefore |A| \neq 0$

$-x^2 - (1 - x^2) \neq 0$

$-x^2 - 1 + x^2 \neq 0$

$-1 \neq 0$

$\therefore$ No restrictions ✓

END OF PAPER

(Deliberately left blank)

(Deliberately left blank)

(Deliberately left blank)